Improving Mathematics Teaching by Developing Mathematical Thinking

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Abstract

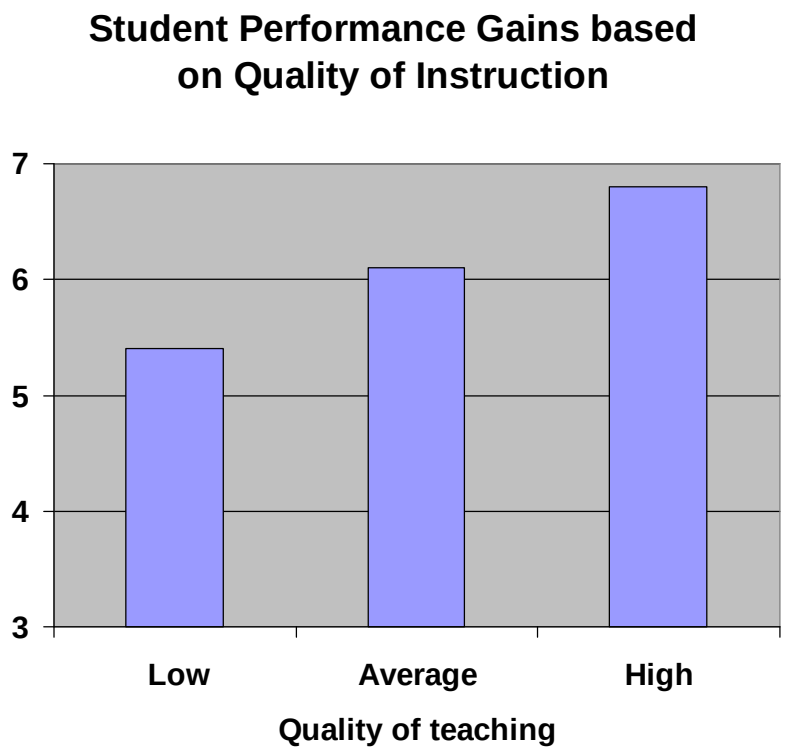
- Infusing ideas from the U.S., the Netherlands, and the Japanese has helped us frame a professional development model that increases teachers' knowledge of mathematics, changes how they think about and teach mathematics, and improves students' thinking of and achievement in mathematics.

"to produce genuine understanding in students, teaching must be guided by detailed research-based knowledge about the ways that students think about particular mathematical ideas. Such knowledge is critical in selecting and creating instructional tasks, asking appropriate questions, guiding classroom discussions, adapting instruction to students' needs, understanding students' reasoning, assessing students' learning progress, and diagnosing and remediating students' learning difficulties" (Battista, 2006, TCM)

Teaching for Understanding

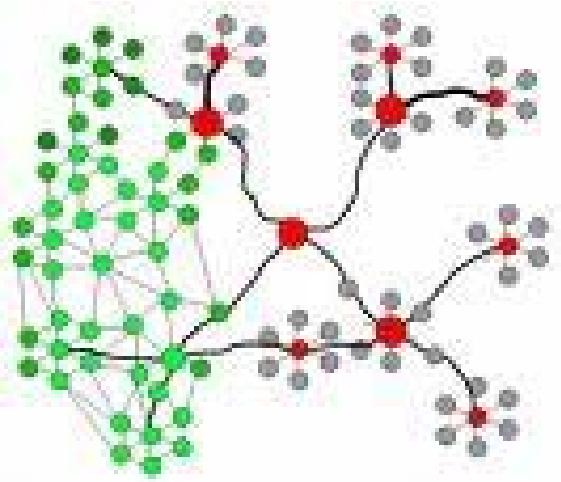
- Connecting related pieces of knowledge together
- Knowing ***how*** to do something and ***why***
- Knowing how . . .
  - Enables us to flexibly use procedures and understand the relationship of these procedures within the structure of mathematics
- Knowing why . . .
  - Enables us to use concepts flexibility, extend our knowledge to new situations, and connect it to the world outside of school

Level of student performance on scales for students who experience low, average, and high 'intellectual quality' instruction (N = 24 schools)



- For every 0.1 increase on the performance scale, a student (on average) will increase 2.1 percentile points on conventional scores (e.g., NAEP, ITBS).
- In other words, if one student was placed in a low quality teaching environment and a student of similar ability (based on NAEP scores) was placed in a high quality teaching environment, the first student would place in the 30th percentile and the other student would be in the 60th percentile.

Teaching for Understanding



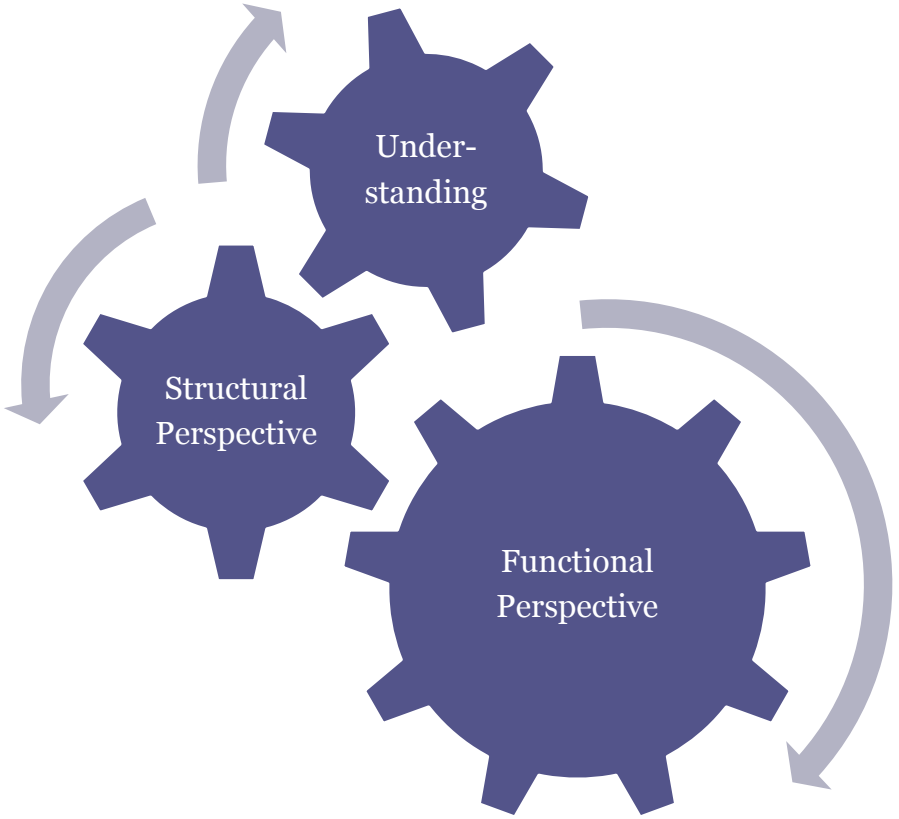
- Structural Perspective
  - Knowledge is structured through web-like/ hierarchal connections.
  - Mental representation as part of a network of representations.
  - Claim: The stronger and the greater number of connections there are in this complex structure, the higher degree of understanding there becomes.

Teaching for Understanding

- Functional Perspective
  - Maintains that students need to actively integrate incoming information with existing knowledge through social interactions.
  - So, students must interact and share knowledge with others.
  - Claim: By being in situations in which students are communicating with others, they build lasting and coherent concepts and skills, which leads to an increased degree understanding.

What does this mean for teaching?

- From a functional perspective, we focus on providing the types of tasks and activities that place students in situations where, through articulation, they are able to reflect on how they solve problems and construct relationships – a structural perspective.



Realistic Mathematics Education

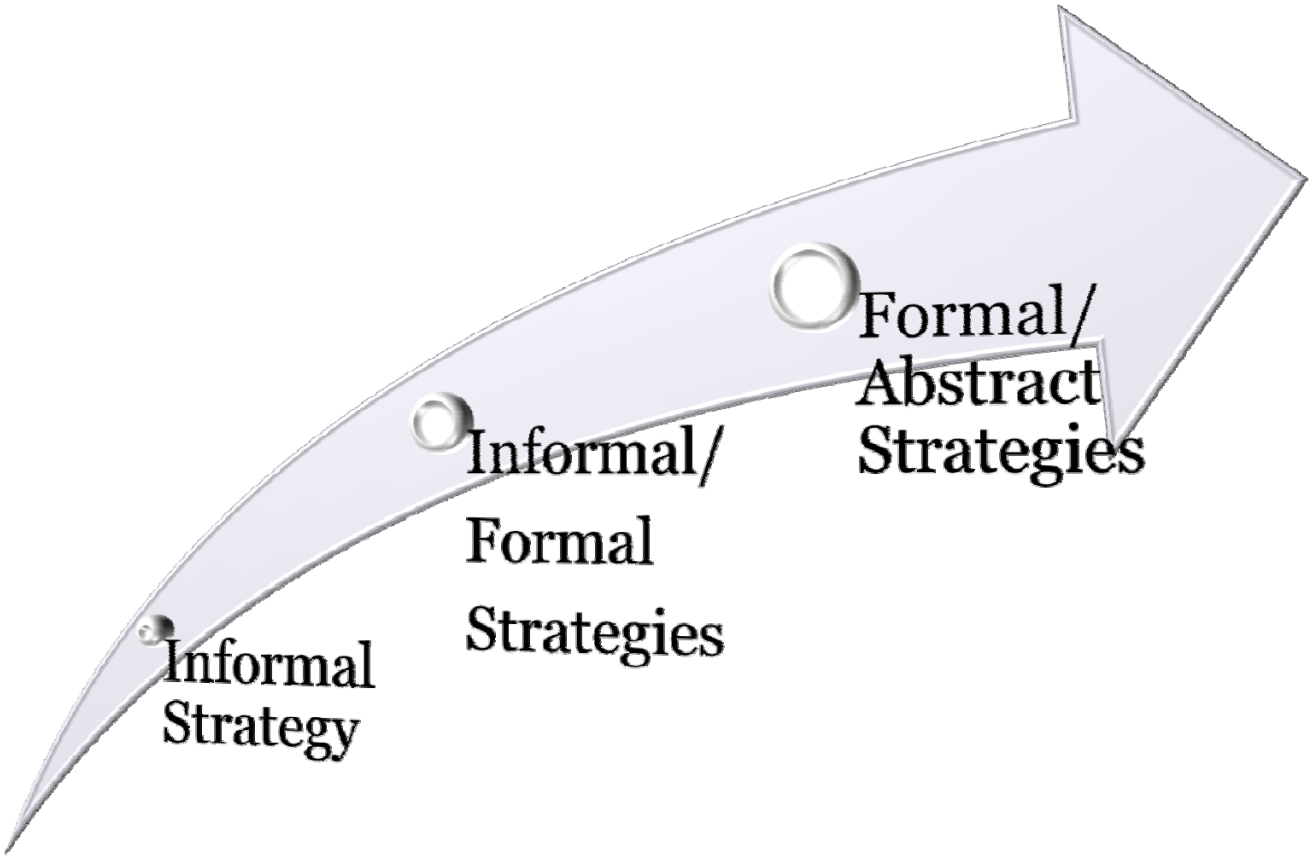
1. Guided reinvention

2. Mathematizing

Guided Reinvention

- Is the process of first allowing students to develop informal strategies for solving problems, and then, by critically examining those strategies, encouraging students to develop more sophisticated, formal, conventional and abstract strategies and algorithms.
- Students are encouraged to make connections between existing knowledge (informal ideas) and new knowledge (more formal mathematical ideas) – structural aspect of understanding.

Guided Reinvention



Guided Reinvention Summary

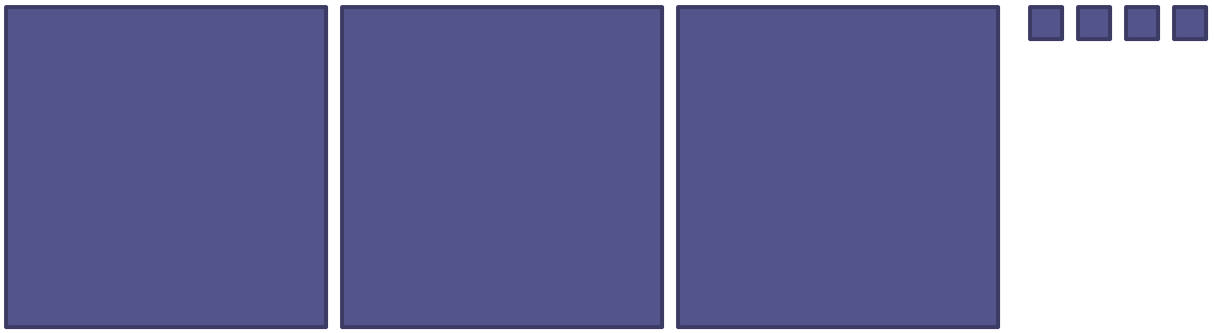
- "By thinking and talking about similarities and differences between arithmetic procedures, students can construct relationships between them. … the instructional goal is not necessarily to inform one procedure by the other but, rather, to help students build a coherent mental network in which all pieces are joined to others with multiple links."

Example 1

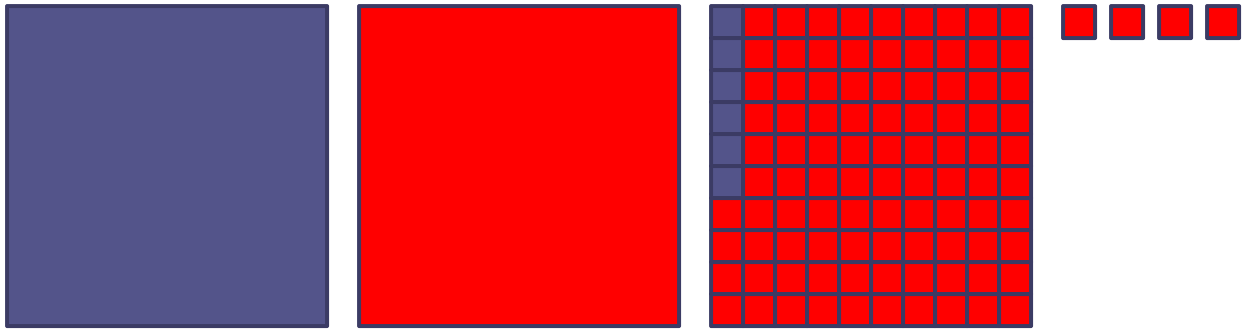
Subtraction

The market was at 304 points last night. This morning it is at 198. How many points did the market go down?

Visual Representation



Visual Representation



100 6

106

Decomposing *using Tree Diagram Notation*

Decomposing *using Place Value*

Compensating

Compensating *using the Open Number Line*

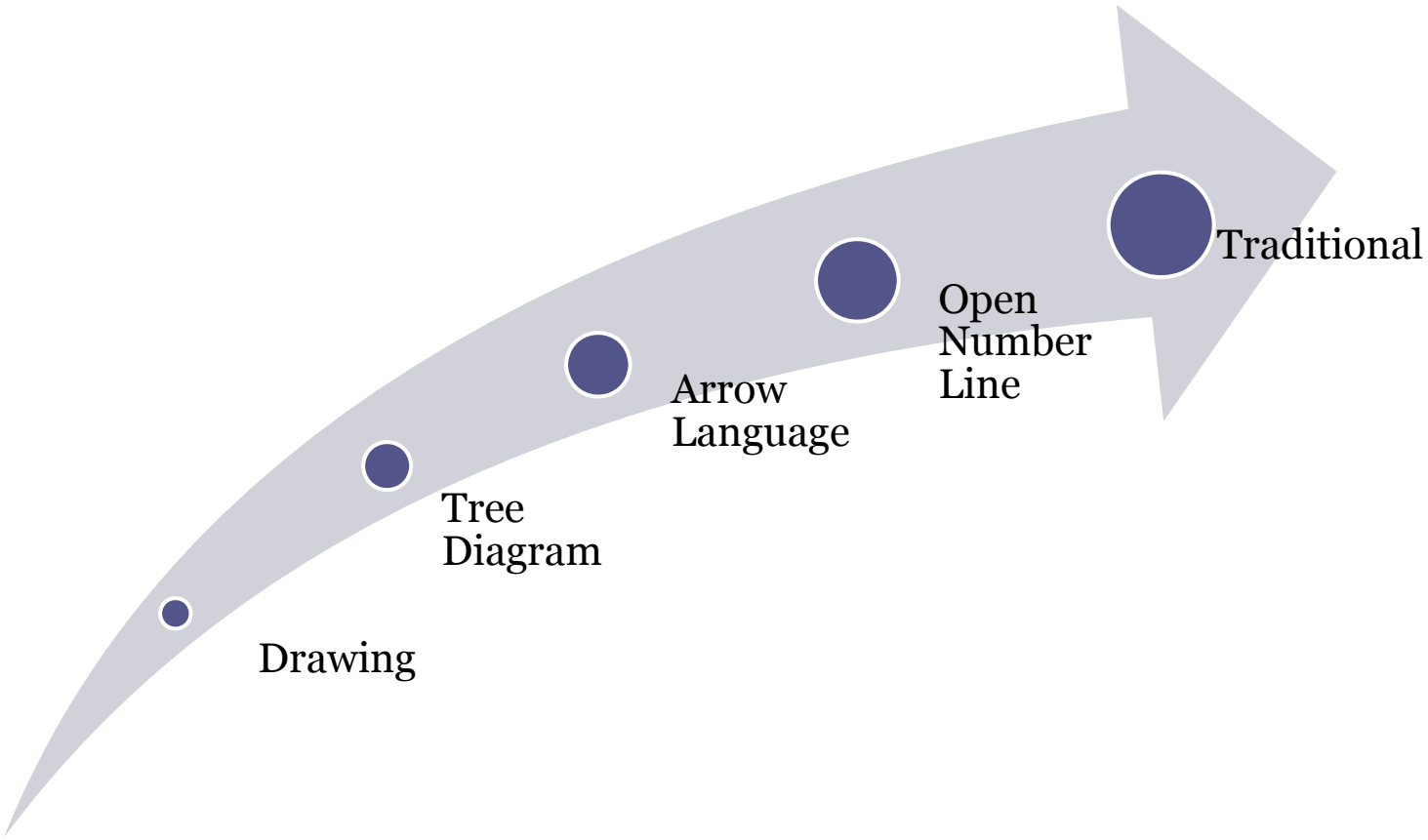
Decomposing *using the Open Number Line*

Decomposing *using Arrow Language Notation*

Thinking of Subtraction as Distance

Traditional Algorithm

Learning Progressions for Subtraction



Mathematizing

Mathematizing

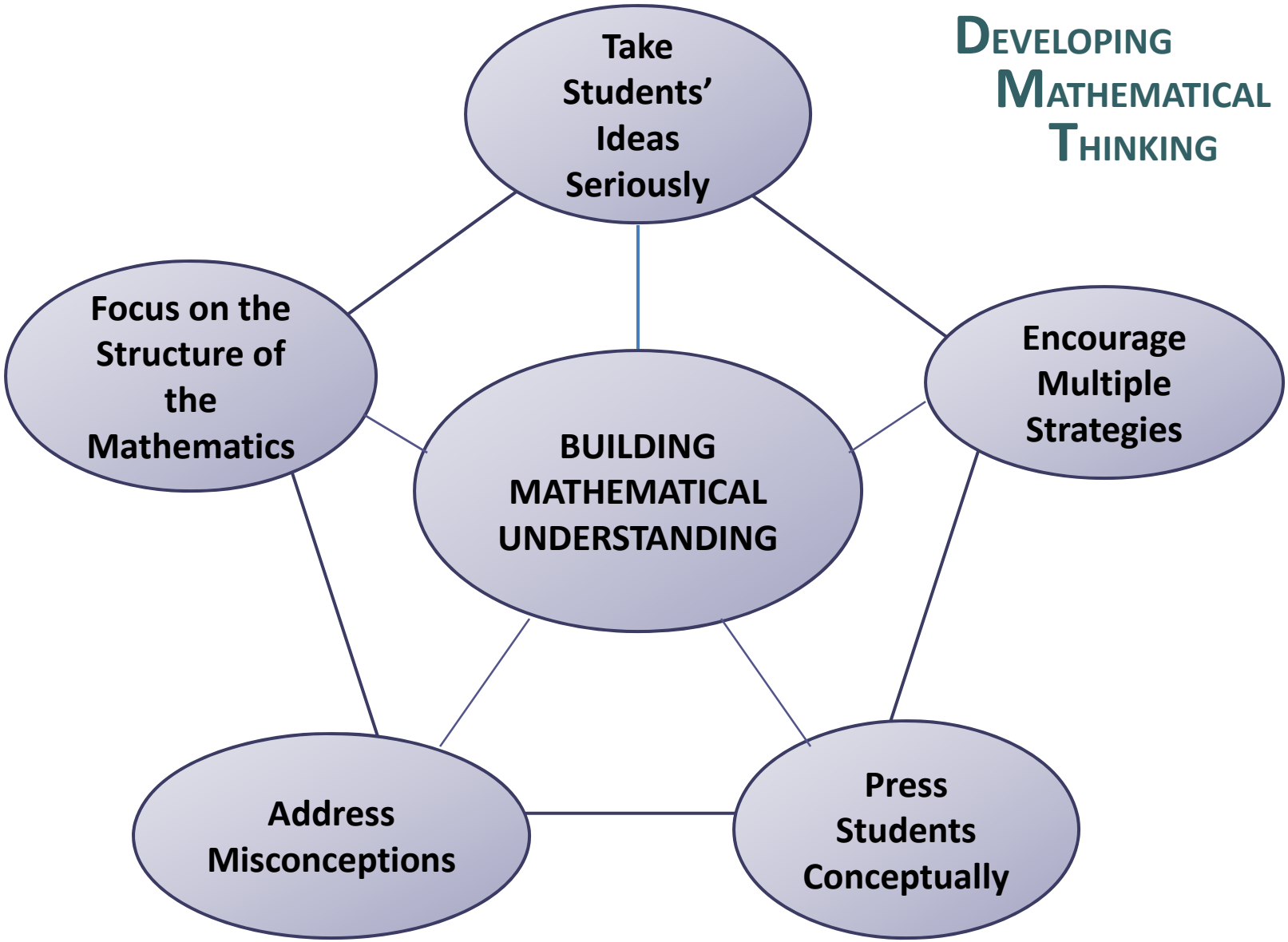
- Horizontal Mathematization
  - Occurs when students represent a contextualized problem mathematically in order to find a solution strategy.
- Vertical Mathematization
  - Involves taking the mathematical matter to a higher level, and occurs when students make their representations and strategies objects of mathematical examination.

Using Mathematizing to DMT

- Horizontal Mathematization
  - Focus on the inherent structure of the mathematical ideas that are emerging
  - Find students' misconceptions and press students to address them conceptually
- Vertical Mathematization
  - Connect students' informal strategies, many of which may be developed outside of school, with more formal mathematical ideas
  - Provide opportunities for students to develop relationships, extend and apply their ideas, articulate what they know, and reflect on what they know and need to know

DMT Ideas

Developing Mathematical Thinking



Taking Students' Ideas Seriously – Centricity

- It is the idea that mathematics pedagogy should be built on the experiences of the student;
- Takes the perspective of positioning students within the context of their own lives, experiences, and cultures;
- Centers students in the process of knowledge acquisition;
- Provides students with the opportunities to solve problems using their experiences.

Mistakes and Misconceptions

Structure of Measurement

- Units
- Zero
- Transitivity
- Conservation

Units

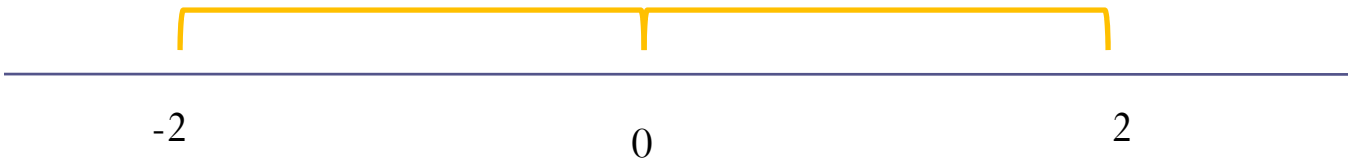
- **Iteration** - consistently repeating a unit with no gaps or overlaps to create a whole
- **Partitioning** - 'splitting' a whole into consistently sized units of measure

- **Equivalence** - consistency and precision
- **Unit-based reasoning** - What unit should be used? What attribute is to be measured? What will my measurement tell me? Proportional relationships (e.g. 1 foot is 12 inches and 24 half-inches) Quantitative Reasoning*
- **Accumulation** - the number of units used constitute the 'measure' (related to the Cardinal Principle of counting)

Zero

- **Origin**-'Zero' units of measure indicates the starting point for the measurement. *Zero is arbitrary* and can 'begin' at any place on the object. (also referred to as the *zero-point*)
- **Absolute value**-distance from zero (most common meaning)

$$|2| = |-2|$$



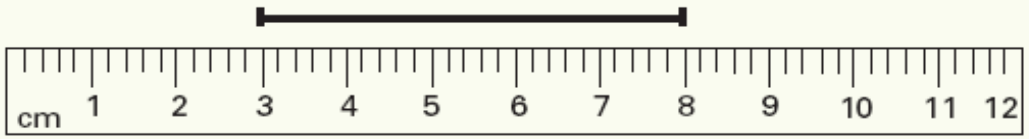
Lehrer, 2003

When students don't understand zero….

Table 1

Percentages of Students in Grades 3 and 7 Responding to an Item of the National Assessment of Educational Progress

Table 5.3
Rulers

Item	Percent Responding[a] Grade 3	Grade 7
 How long is this line segment?[b]		
3 cm	4	1
5 cm *	14	49
6 cm	31	37
8 cm	30	9
11 cm	6	2
I don't know.	15	2

[a] The response rate was .80 for grade 3 and .97 for grade 7.

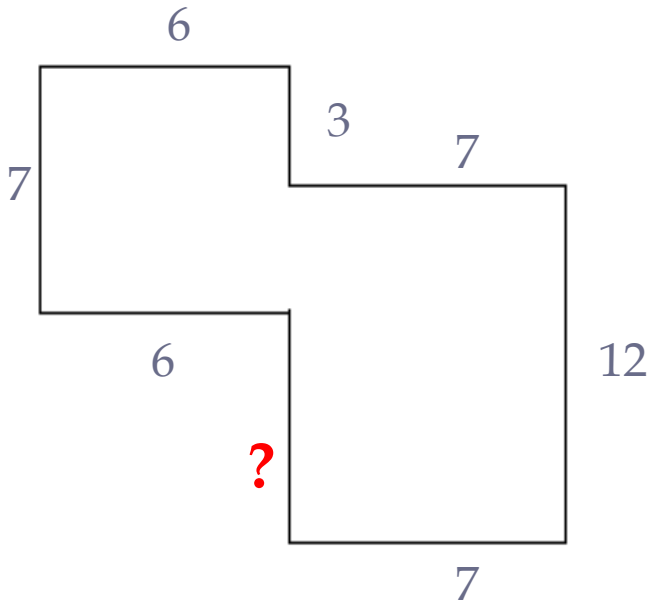
[b] An actual centimeter ruler was pictured.

* Indicates correct response.

Kamii, 2006

Transitivity

- **Comparisons** - using known measurements to find unknown measurements
- **Quantitative*** - using a comparison of an attribute to a unit. The *number* of units is used to solve the problem. ("This pencil is 4 units long")
- **Qualitative** - making general comparisons between objects' common attribute to solve a problem ("This pencil is the same as this side of the rectangle.")

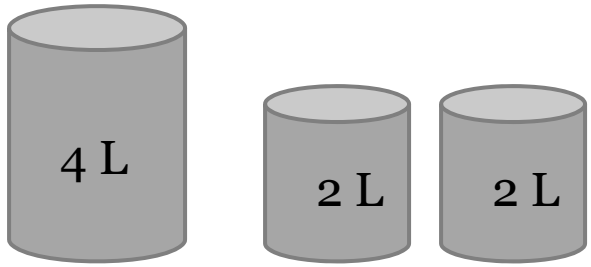


* If a=b and b=c, then a=c (One view of *transitivity*) The *transitive property* is about the relationships between a, b, and c…not necessarily relationships of equality.

Battista, 2006; Lehrer, 2003

Conservation

- Measurements can stay the same even when the object is moved, decomposed, or rearranged.



DMT Professional Development

Professional Development Plan

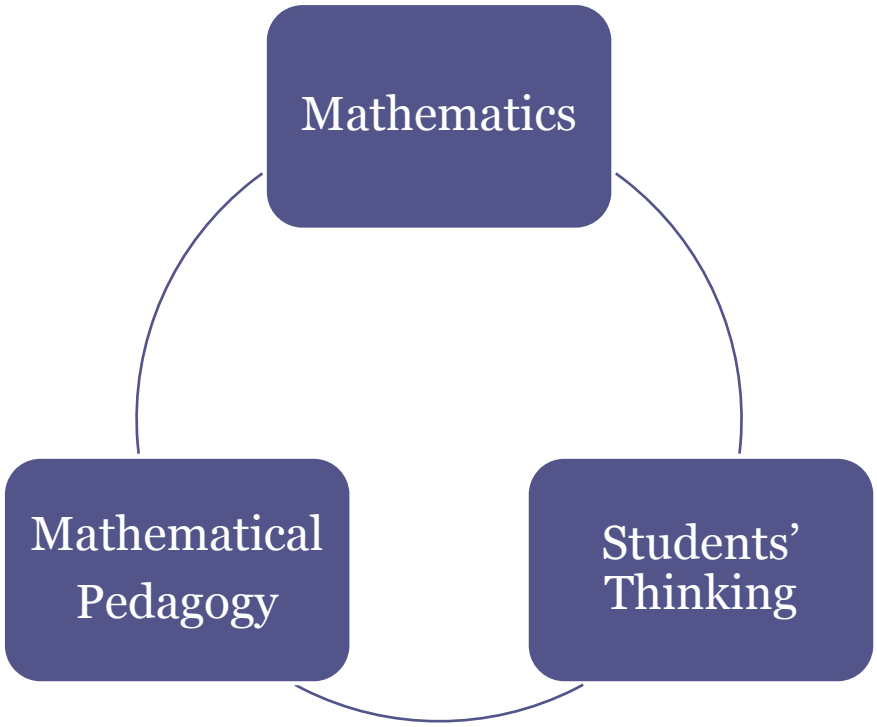
Professional Development	***DEVELOPING MATHEMATICAL THINKING***		
	Year 1	**Year 2**	**Year 3**
Focus Area	*Number & Algebra*	*Measurement & Geometry*	*Probability & Statistics*
Summer PD 5 Days (45 hours) In-depth topics			
Ongoing PD 18 days Lesson Study (**4 X semester**) Observations (**monthly**) Demonstrations (**monthly**)			

Lesson Study

- Identify long term goal and collectively plan lessons that will help reach that goal.
- Teach one lesson while others observe and collect data on student thinking, learning, etc.
- Share and analyze the data to determine whether the goals were met and what improvements need to be made.
- Refine and re-teach the lesson, study it again.

Professional Development Goals

- Teachers must know not only the formalism of mathematics but the informal and generative process of students' mathematical ideas and how to encourage fluid growth of these ideas.



Professional Development Goals

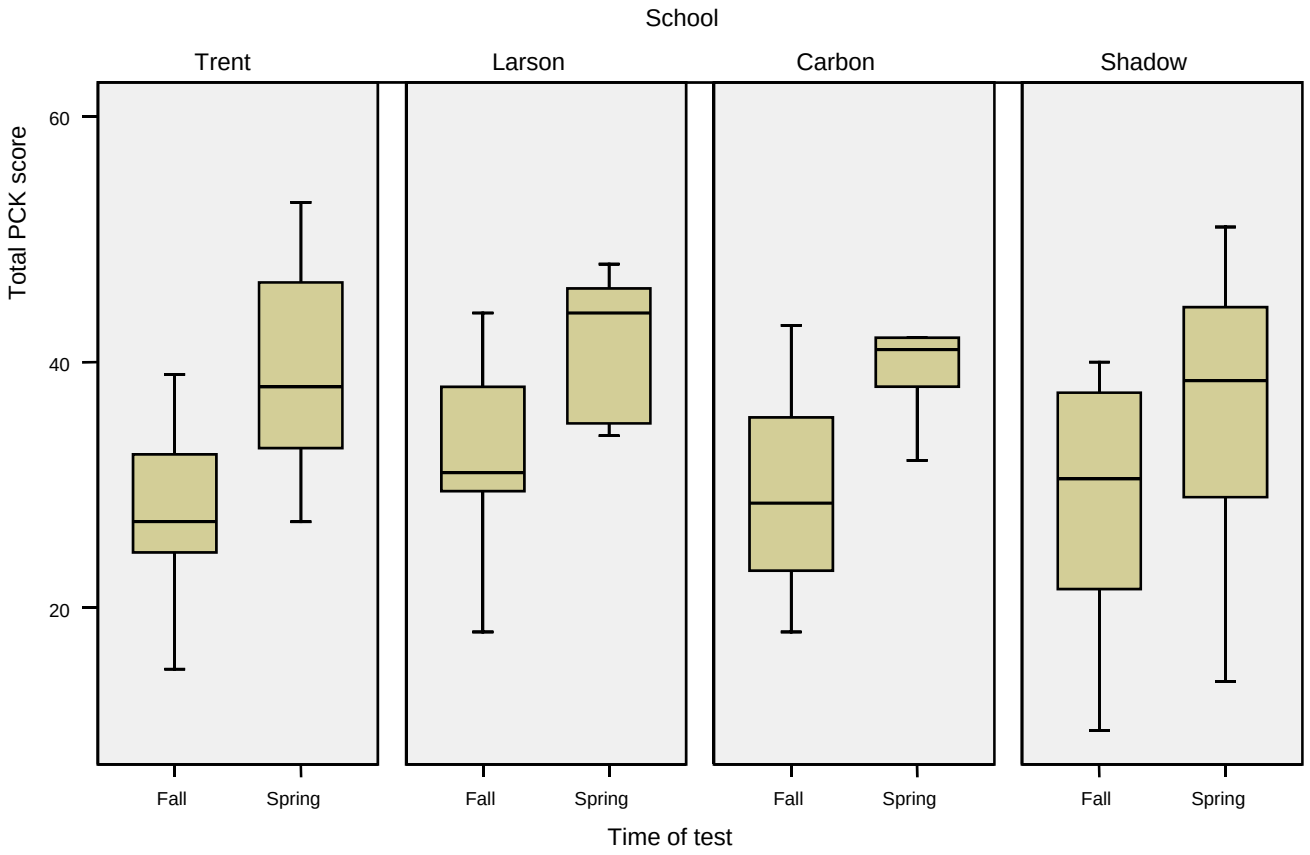
- The professional development goals are to encourage elementary teachers to provide instruction:
  - That uses **reasoning** to make sense of problems;
  - That moves from **informal** idea to more **formal** and abstract ones;
  - That uses **articulation** of one's ideas and conjectures as a focal point to improved understanding.

DMT

A Three Year Study

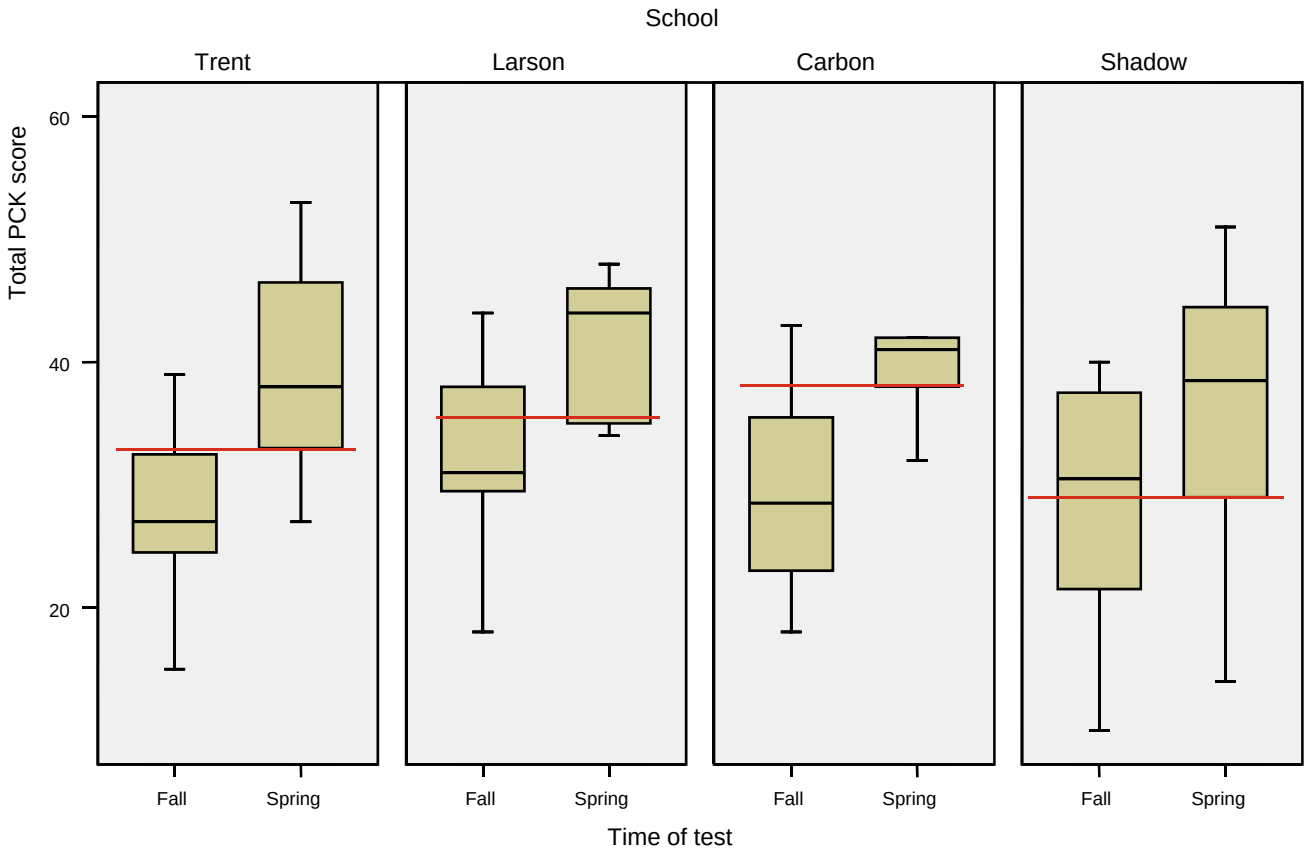
Teacher Knowledge Gain

- August mean 29.74 (SD 7.8)
- March mean 39.16 (SD 7.8, 0.000)

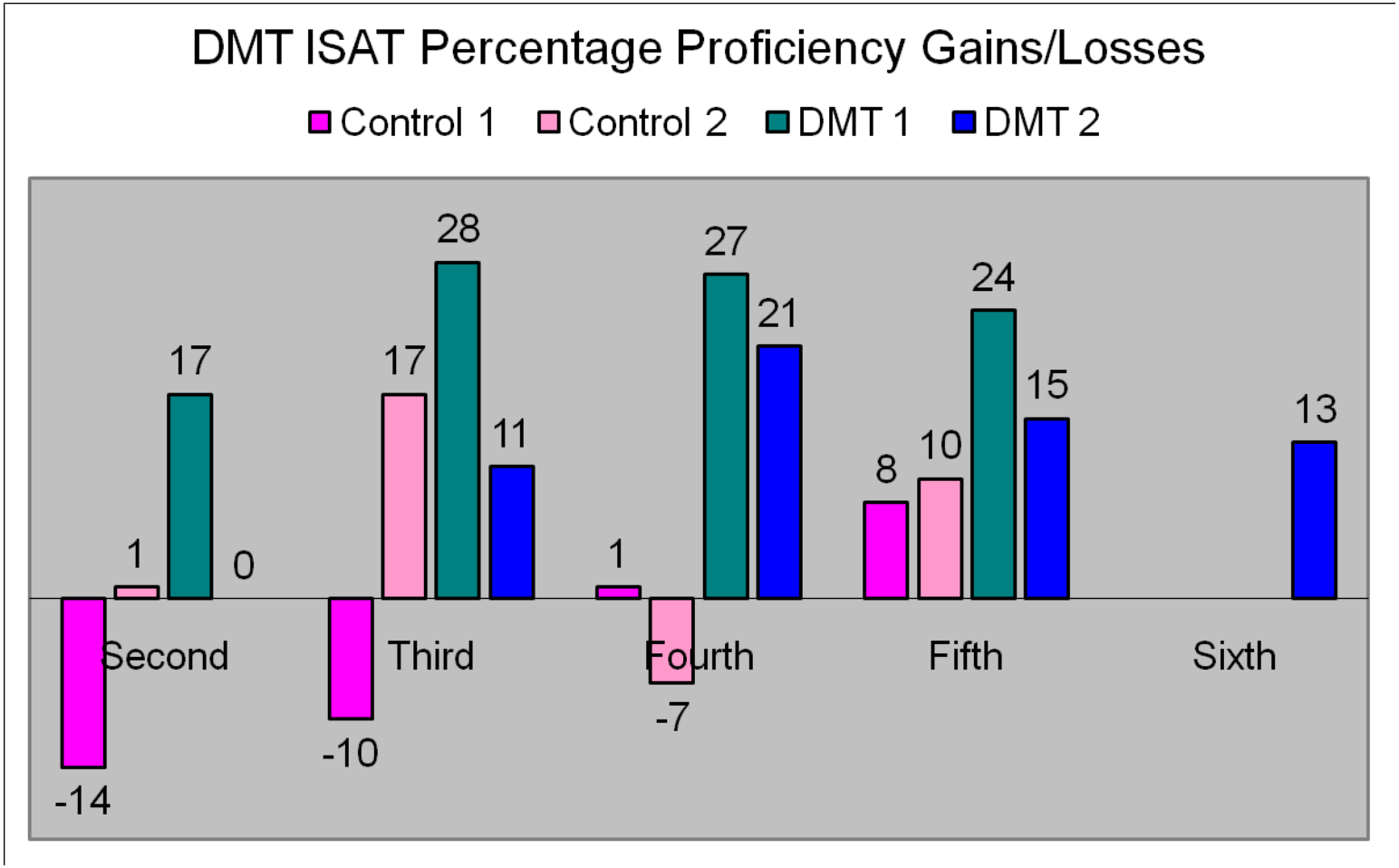


Teacher Knowledge Gain

- In many of the schools, what only a handful of teachers knew before the institute, many of them knew six months later.



ISAT Gains

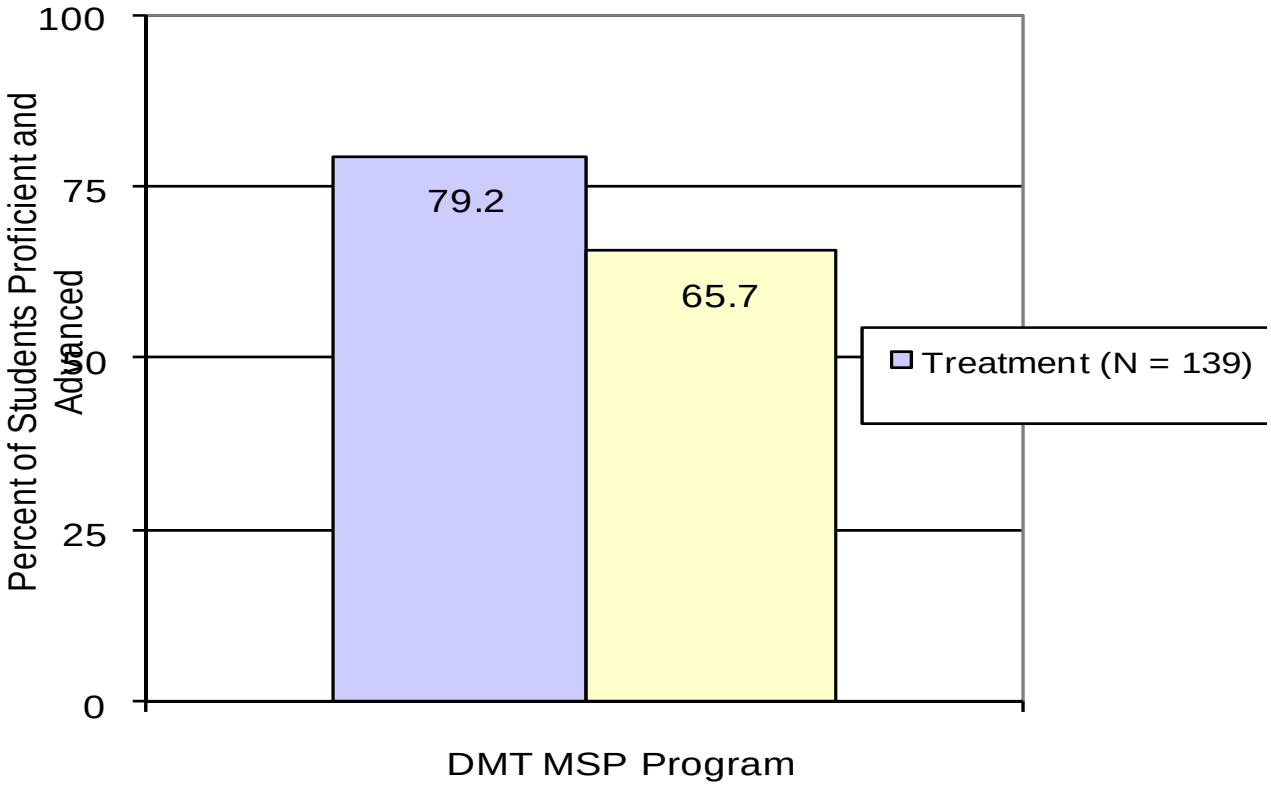

DMT ISAT Percentage Proficiency Gains/Losses
Control 1
Control 2
DMT 1
DMT 2
-14
1
17
0
-10
17
28
11
1
-7
27
21
8
10
24
15
13
Second
Third
Fourth
Fifth
Sixth

Characteristics of Students of DMT Teachers and Stratified Samples of Comparison Students: 2006-2007 School Year

Group	Number of Scores	Percentage			
		Below Basic	Basic	Proficient	Advanced
DMT Students	139	3.6	17.3	50.4	28.8
Comparison Students	318	9.7	24.5	46.2	19.5

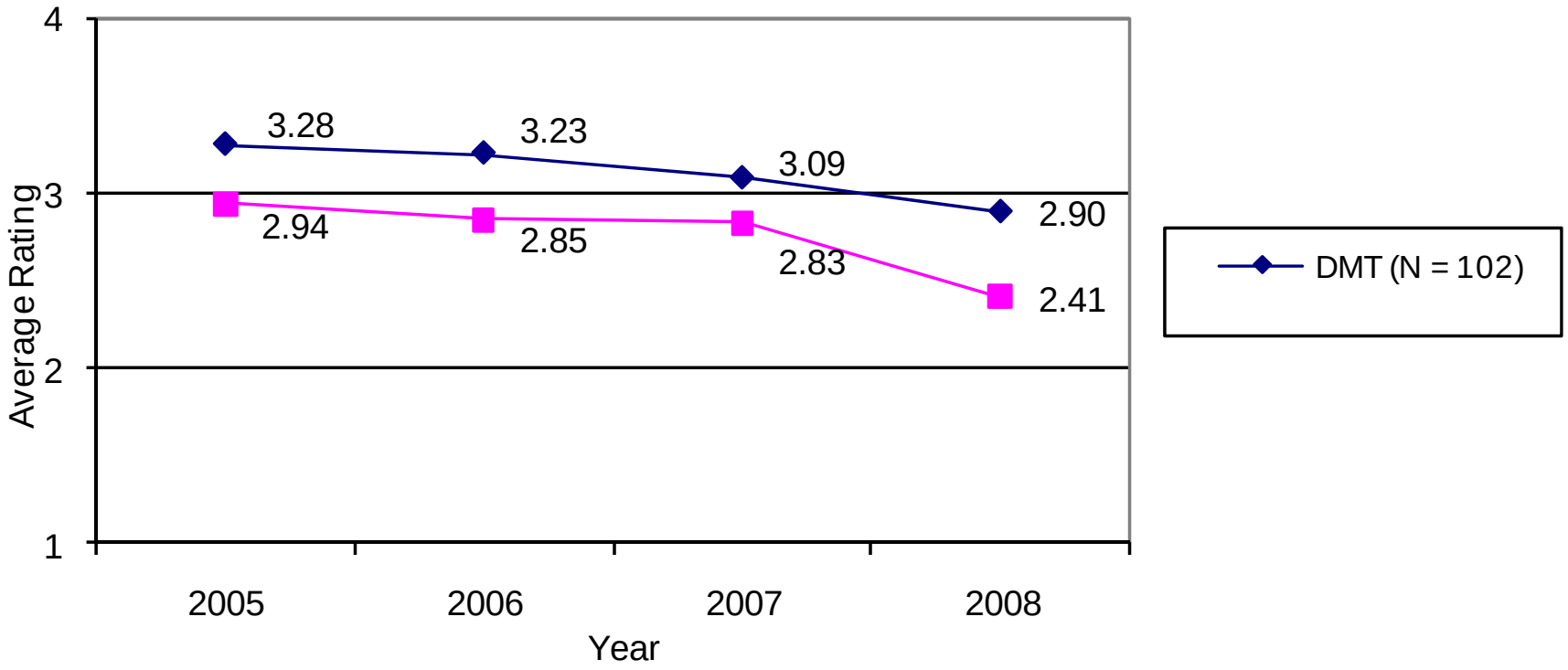
RMC, 2008

Percentages of Students That Scored Proficient and Advanced on the Spring 2007 ISAT Math Test, Broken Down by Teacher Participation in DMT



RMC, 2008

Comparison of Student ISAT Math Scores Over Four Years for DMT Participating Teachers and Comparison Teachers



RMC, 2008

IDMT
Initiative for
Developing Mathematical Thinking

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